SIMPLE FORMULAE FOR PROCESSING OF THE RESULTS OF A ONE-POINT-BEND TEST

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ABSTRACT

The influence of the geometry of bend specimen on the shape and amplitude of dynamic stress intensity factor (DSIF) variation in time during a one-point bend impact test has been investigated numerically. Specimens with the relative length \( \frac{3}{6} \) and relative crack depth \( 0.3 \ldots 0.7 \) have been considered. Utilizing classical Timoshenko’s approach, the process of specimen interaction with the striker has been modelled by the Volterra integral equation of the second kind with respect to unknown contact force. Both simplified analytical and exact numerical solutions of this equation have been obtained. When the impact force is known, DSIF is calculated using modal superposition technique proposed previously by the author. It has been shown that for specimens with the relative length \( 3 \ldots 4.5 \) DSIF grows almost linearly to its maximum value. For all configurations of the specimen considered this value is practically independent of the specimen relative length and increases with increasing of the relative crack depth. Simple formulae for estimation of the maximum DSIF value and the moment of time when it is reached have been proposed.

KEYWORDS

Instrumented impact test, one-point-bend test, dynamic stress intensity factor, modal superposition, numerical methods, DSIFcalc program

NOTATION

- \( \alpha \) - crack length
- \( B \) - thickness of a specimen
- \( E \) - Young’s modulus of the specimen material
- \( F(t) \) - force measured on striker
- \( K_I(t), \text{DSIF} \) - dynamic stress intensity factor
- \( K_{\max}, T_{\max} \) - The maximum DSIF value for a test and the time when it is reached.
INTRODUCTION

Instrumented impact fracture testing of the precracked beam specimens is widely used for determination of the dynamic fracture toughness of brittle materials. Experimental results [1] and numerical analysis [2] have shown, that during the early stage of loading the specimen may loose contact with the supports. At this time bending of the specimen is not three-point but one-point one, i.e., it is caused by tup loading and specimen inertia only. That is the reason why such a type of impact test, in which crack growth is initiated during this initial period of loading\(^1\), has been called one-point bend test (1PB) [1, 3].

Thus, 1PB stage is an inevitable initial stage of any impact bend test. During this stage transient dynamic effects dominate in the specimen response so \(K_i(t)\) is not proportional to \(F(t)\). Thus, to evaluate \(K_i(t)\) precisely, some kind of dynamic analysis of the test should be performed. Although a 'brute-force' finite element analysis (FEA) can be used for this purpose, more smart and cheap methods for DSIF calculation have been proposed [4–8]. If \(F(t)\) is known, some of these methods allow to obtain as accurate values of \(K_i(t)\) as the full-scale FEA could give. However, it does not solve all problems related with 1PB test.

Contrary to the conventional three-point bend test, the 'DSIF-capacity' of the 'native' 1PB test (means a test performed using a supportless specimen) is limited. In such a test DSIF can grow only until the velocity of the specimen centre of mass is lower than the striker velocity.

\(^1\)This is the case for high velocity testing of brittle plastics, ceramics or hardened steels.
When the accelerated specimen finally loses contact with the striker, \( K_I(t) \) drops to zero. Thus, to plan a 1PB test properly, an experimenter needs the preliminary information about:

- The maximum DSIF value, \( K_{\text{max}} \), that could be reached for a particular geometry of the specimen, its material properties and impact velocity.
- The duration of the ‘DSIF-growth’ stage of the planned test \( T_{\text{max}} \). This value allows to choose the proper probing rate for the registration equipment and to estimate DSIF growth rate during a test, which is usually associated with the rate of material deformation in the process zone.

The aims of this work were:

1. To propose a simple yet sufficiently accurate model of the 1PB test.
2. To investigate numerically the influence of the specimen geometry parameters \( \gamma \) and \( \lambda \) on the shape and amplitude of DSIF.
3. To propose simple approximate formulae for estimation of \( K_{\text{max}} \) and \( T_{\text{max}} \).

THEORY

Initial Assumptions

The following assumptions have been used to model the specimen interaction with a striker:

- specimen material is linearly elastic;
- specimen-striker interaction is modeled by point force \( F(t) \);
- the striker is perfectly stiff and has constant curvature \( r_F \) of the tip;
- contact between the specimen and striker is described within quasi-static Hertz’s approach;
- specimen-striker contact stiffness can be linearized.

All these assumptions are usually satisfied when brittle polymers are tested. The assumption that the striker is perfectly stiff is an idealization of the real situation. It gives us, however, an occasion to consider the behaviour of the specimen in the most extreme situation when the amplitude of loading and DSIF will be the largest.

Some additional analysis has been performed to derive a simple formula for estimation of the linearized contact stiffness between the specimen and stiff cylindrical indenter that models a striker. As a result of mixed analytical analysis and 3D finite element calculations using commercial finite element program ADINA, the following approximate relation between the depth of the indenter penetration \( \delta \) into the specimen surface and total contact force \( P \) has been derived

\[
\delta = \frac{P}{\Phi(B, W, r, \nu)B^E\left(\ln \frac{\pi BW^2E}{rP}\right)}
\]

(1)

where \( \Phi(B, W, r, \nu) = [1 - 0.15\nu/W][3.038 + (1.071B/W + 0.361)\nu^2] \), \( r \) is the radius of curvature of the indenter [9]. For \( P/(BWE) \leq 0.0005 \), \( B/W = 0.1 \ldots 1.0 \), \( r/W = 0.1 \ldots 0.5 \),
\( \nu \leq 0.4 \) the error of data obtained using Eq. (1) remains less than 5% when compared with corresponding FEA results. It is worth to note that for \( P/(BWE) \leq 0.0005 \) dependence of \( \delta \) on contact force in Eq. (1) is nearly linear. Thus, its linearization is natural and does not lead to a large error. As a result of such an operation the following formula has been obtained for linearized contact stiffness

\[
k = \frac{\Phi(B, W, \tau, \nu)EB}{\ln \left( \frac{\pi BW^2E}{2P_{\text{max}}} \right) + \frac{1}{2}}
\]

where \( P_{\text{max}} \) is the maximum contact force [9].

**Governing Equations**

![Figure 1: Two-dimensional model of the test specimen](image)

Let's consider the 2D model of the specimen (see Fig. 1). Utilizing classic Timoshenko’s approach, one may assume that the forces experienced by the striker is equal to corresponding linearized contact force, so

\[
F(t) = k\delta_F
\]

The depth of specimen surface penetration by striker \( \delta_F \) can be defined as

\[
\delta_F = \begin{cases} 
\mathbf{u}_{\text{str}}(t) - \mathbf{u}_F(t) & \mathbf{u}_{\text{str}}(t) \geq \mathbf{u}_F(t) \\
0 & \mathbf{u}_{\text{str}}(t) < \mathbf{u}_F(t)
\end{cases}
\]

The following equation can be obtained easily for the striker tip position

\[
\mathbf{u}_{\text{str}}(t) = v_0 t - \frac{1}{M} \int_0^t F(\tau) |t - \tau| \, d\tau
\]

Here the effect of increasing the striker velocity during impact due to gravitation (that is the case for a drop weight machine) is neglected.

Specimen displacement in the striker/specimen contact point consists of two parts. The first is \( x \)-component of specimen movement as a rigid body \( \mathbf{u}_F^{(rb)}(t) \) that can be expressed as

\[
\mathbf{u}_F^{(rb)}(t) = \frac{1}{m} \int_0^t F(\tau) |t - \tau| \, d\tau
\]

The second part of \( \mathbf{u}_F(t) \) is the displacement due to specimen bending \( \mathbf{u}_F^{(bend)}(t) \). Using modal superposition method, it can be expressed as

\[
\mathbf{u}_F^{(bend)}(t) = \sum_{i=1}^{N_F} \frac{[\psi_i(t)]^2}{\omega_i} \int_0^t F(\tau) \sin(\omega_i(t - \tau)) \, d\tau
\]

if \( N_F \) eigenfrequencies \( \omega_i \) and normalized nontrivial symmetrical\(^2\) eigenmodes \( \psi_i(x, y) \) for the unsupported specimen are taken into account.

Combining Eq. (3) with Eqs. (4)-(7) results in

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\(^2\)Non-symmetrical eigenmodes are ignored because they do not cause crack opening and non-zero SIF arising
\[ \frac{F(t)}{k} = v_0 t - \int_0^t F(\tau) \left( \frac{t - \tau}{m} + \sum_{i=1}^{N_r} \frac{|\psi_i|^2}{\omega_i^2} \sin(\omega_i(t - \tau)) \right) d\tau \]  

(8)

This equation is valid for \( u_{str}(t) \geq u_F(t) \), otherwise \( F(t) \equiv 0 \). When \( F(t) \) is known, \( K_f(t) \) can be calculated using the formula

\[ K_f(t) = \sum_{i=1}^{N_k} \int_0^t F(\tau) \left( \sum_{i=1}^{N_r} \eta_i^{(1)}(t) \cos(\omega_i(t - \tau)) \right) d\tau \]  

(9)

where physical sense of coefficients \( k_s^{(1)}, \eta_i^{(1)} \) has been explained and their as well as \( \omega_i \) numerical values for \( i = 1 \ldots 6 \) have been determined previously for 2D (plane stress) [6] and 3D [10] models of the specimen. Eq. (9) can be considered as a generalization of the Kishimoto’s formula [11]. When \( F(t) \) is approximated piece-wise linearly or by Fourier series, integration in Eq. (9) can be performed analytically and simple formulas in a closed form can be obtained [6]. Freeware program DSIFcalc [12] can be used for related calculations.

**Analytical Solution**

Eq. (8) is the Volterra integral equation of the second kind with convolution type kernel. Such an equation can be solved using Laplace transform technique. Combining transformed Eqs. (8) and (9) one can obtain the following equation for the transform \( \mathcal{L}[K_f(t)] \) of \( K_f(t) \)

\[ \mathcal{L}[K_f(t)](s) = v_0 kk_s^{(1)} \sum_{i=1}^{N_k} \eta_i^{(1)} \frac{k_s^{(1)} \omega_i^2}{\Omega^2 + s^2} \left( \frac{\Omega^2}{1 + \sum_{i=1}^{N_r} \frac{k_s^{(1)} \omega_i^2}{s^2 + \omega_i^2}} \right) . \]  

(10)

If \( \eta_i^{(1)}, \omega_i \) and \( |\psi_i|^2 \) are known, the inverse Laplace transform of this equation can be obtained numerically. However, for \( N_F = 1 \) an analytical solution can be obtained too. In the simplest case \( (N_F = N_K = 1) \) the resulting formula is

\[ K_f(t) = \frac{v_0 k_s^{(1)} \omega_i^2 \eta_i^{(1)}}{4C_1 C_2} \left( \frac{\sin(C_1 - C_2)t}{C_1 - C_2} - \frac{\sin(C_1 + C_2)t}{C_1 + C_2} \right) \]  

(11)

where \( C_1 = 1/2\sqrt{k_s^{(1)} |\psi_i|^2 + (\Omega + \omega_i)^2} \), \( C_2 = 1/2\sqrt{k_s^{(1)} |\psi_i|^2 + (\Omega - \omega_i)^2} \).

**Numerical Solution**

The analytical solution presented above assumes that the striker is in permanent contact with the specimen during impact. This assumption contradicts with the results of both experimental observations [1] and numerical simulations [2, 7]. Fortunately, Eq. (8) can be easily solved numerically even in more general case when the specimen-striker contact is not permanent. To perform computations in such a way, the free program DSIFcalc [12] can be used again.
RESULTS

Modal parameters of the specimen (namely, $n_i^{(1)}$, $\omega_i$ and $\psi_i$, $i = 1 \ldots 6$) have been determined by 2D FEA using ADINA for $\gamma = 3, 3.1, \ldots 6$, $\lambda = 0.3, 0.4, \ldots 0.7$, $\nu = 0.2, 0.25, \ldots 0.4$. These data have been used to determine the modal parameters for any particular configuration of the specimen by interpolation.

To check the accuracy of both analytical and numerical solutions, the 1PB test reported by Böhme and Kalthoff [1] has been modelled. In this experiment a large-scale $(W = 0.1 \text{ m}, B = 0.01 \text{ m}, \gamma = 4.12, \lambda = 0.3)$ Araldite B (E = 3.38 GPa, $\rho = 1260 \text{ kg/m}^3$, $\nu = 0.33$) specimen has been tested on a drop-weight machine ($M = 4.9 \text{ kg}, \tau_F/W = 0.08$) with $v_0 = 1 \text{ m/s}$. Using the maximum impact force (about 1.5 kN) registered experimentally, the contact stiffness between the specimen and the striker has been estimated as 8.5 MN/m using Eq. (2). Experimental and theoretical DSIF-time curves are compared in Fig. 2. Even the simplest 1-mode analytical formula (11) gives reasonable (although a bit underestimated) $K_i(t)$. Increasing the number of eigenmodes taken into account in calculations improves the accuracy of the results considerably (of course, partial coincidence between the experimental $K_i(t)$ and the curve obtained from the analytical solution for $N_F = 1, N_K = 3$ is accidental).

To investigate $K_i(t)$ dependence on the specimen geometry parameters, 1PB test has been simulated numerically for $\gamma = 3, 3.1, \ldots 6$, $\lambda = 0.3, 0.4, \ldots 0.7$, $\nu = 0.3$ using DSIFcalc program. This time, contrary to the analytical solution presented previously, the possibility of loss of the contact between the specimen and the striker has been taken into account. Striker mass was supposed to be infinite in all these simulations. Together with the initial assumption that striker’s stiffness is infinite, this simplification allows us to consider the results of computations as the 'upper bond' of the real-life situation. Thus, the amplitude of DSIF obtained in these simulations should be considered as the upper limit for the real tests.

All calculations have been performed for $\tau_F/W = 0.25$, because the preliminary analysis has shown that for $\tau_F/W = 0.1 \ldots 0.5$ variation of the radius of the striker tip has negligible effect on $K_i(t)$.

Computed nondimensional DSIF $K_i(t)/[v_0\sqrt{WE\rho}]$ are presented in Fig. 3 as 2D functions, which depends on nondimensional time $t\sqrt{E/\rho/W}$ and $\gamma$ or $\lambda$. For each fixed value of $\lambda$, $K_{max}$ values for all specimen configurations are practically independent of $\gamma$ (Fig. 3a). When $\gamma$ is fixed, $K_{max}$ values increase with increasing $\lambda$ (see Fig. 3b). In all cases $K_i(t)$ grows almost linearly up to $K_{max}$ for $\gamma \leq 4.5 \ldots 5.0$. For higher values of $\gamma$, DSIF starts to oscillate before $K_{max}$ is reached (note a ‘wave’ on the surface in Fig. 3a). Perhaps this phenomenon might be the consequence of the assumption about the infinite stiffness of the striker, because such a 'waves' disappear when considerably less stiff contact conditions are considered [13].
The Useful Formulae for Planning a One-Point-Bend Experiment

Values of $K_{\text{max}}$ and $T_{\text{max}}$ can be estimated from Eq. (11). From the condition that the first derivative of DSIF with respect to $t$ is equal to zero, one can obtain the time at which $K_{\text{max}}(t)$ reaches the maximum

$$T_{\text{max}} = \pi / C_1.$$  

(12)

After substitution of this value into Eq. (11) one can obtain

$$K_{\text{max}} = \frac{\nu_0 k \phi^{(1)} \omega_1 \nu^{(1)}_2}{2 \Omega C_2} \sin \left( \frac{\pi C_2}{C_1} \right).$$  

(13)

As any analytical solution, Eqs. (12),(13) show explicitly how $T_{\text{max}}$ and $K_{\text{max}}$ depend on the various test and specimen parameters. However, these formulae cannot be used to predict $T_{\text{max}}$ and $K_{\text{max}}$ directly. They both contain the contact stiffness $k$, which depends on the unknown maximum impact force (see Eq. (2)). Although this dependence is weak (large variations of $P_{\text{max}}$ cause small changes in $k$) it complicates the usage of Eqs. (12),(13).

Simplicity of the shapes of the non-wavy (means obtained for $\gamma \leq 4.5$) DSIF-time curves suggests that their initial parts might be approximated by straight lines (see Fig. 4). In such a case, one may suppose that $K(t)$ starts to grow linearly at $T_{\text{int}} = (W - a) / c_{\text{sh}}$ (where $c_{\text{sh}}$ is the shear wave velocity for the specimen material) and reaches $K_{\text{max}}'$ at $T_{\text{max}}'$. Analysis of the computed DSIF-time curves shows that $K_{\text{max}}'$ is about 98% of $K_{\text{max}}$ for $\lambda = 0.3 \ldots 0.6$ and about 90-95% of $K_{\text{max}}$ for $\lambda = 0.7$. By fitting the numerical data the following approximated formulae have been obtained

$$K_{\text{max}}' = (1.481 + 2.689\lambda)\nu_0 \sqrt{WE\rho}$$  

(14)
The accuracy of these formulae is not high (about 6% for Eq. (14) and about 11% for Eq. (15)) because they were intentionally made as simple as possible.

CONCLUSIONS

An analysis is presented for the DSIF-response of the bend specimen during an 1PB impact test. Simplified analytical and exact numerical solutions of the derived equations have been obtained for the specimens with the relative length $3 \ldots 6$ and relative crack depth $0.3 \ldots 0.7$. It has been shown that for specimens with relative length $3 \ldots 4.5$ DSIF grows almost linearly to its maximum value. For all configurations of the specimen considered this value is practically independent of the specimen relative length and increases with increasing of the relative crack depth. Simple formulae for estimation of the maximum DSIF value and the moment of time when it is reached have been proposed.

References